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# EVOLUTION OF PERTURBATIONS ON THE SURFACE OF A VISCOELASTIC LIQUID

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*The stability of a layer of a viscoelastic liquid on an inclined plane is studied within the framework of the model with a time-dependent "memory" in the presence of surface tension. It is shown analytically and numerically that these flows can be stable or unstable depending on the Reynolds number. Profiles of the free surface are found as functions of the Reynolds and Weber numbers.*

The properties of non-Newtonian viscoelastic fluids called second-order liquids are often described on the basis of the model with a decaying "memory" [1–4]. It is shown [5–7] that the surface of a layer of such a liquid at Reynolds numbers higher than a certain critical value becomes unstable to small long-wave perturbations, and the critical Reynolds number for a viscoelastic liquid is smaller than for a Newtonian one. Dandapat and Gupta [8] studied the process of formation of solitons on the free surface of a film of a second-order liquid moving down an inclined plane at Reynolds numbers higher than the critical value. Using decomposition of the initial equations in powers of the small parameter (the ratio of the layer thickness to the characteristic spatial scale along the layer), they obtained a nonlinear equation for the free-surface shape with account of surface tension. For high Weber numbers characterizing the ratio of surface-tension forces to gravity forces, Dandapat and Gupta [8] studied the steady solutions of this equation using methods of the qualitative theory of ordinary differential equations and analyzed the character of singular points and trajectories in the phase space. In addition, in the approximation of small Weber numbers, small deviations of the free-surface shape from its unperturbed state, and small supercritical values, Dandapat and Gupta [8] reduced the resultant unsteady equation to the Korteweg–de Vries equation, in which the dispersion length increases with increasing effects of viscoelasticity, and conducted a series of numerical calculations using the known finite-difference scheme with jumps, which was used in the first works devoted to numerical solution of this equation. The formation of solitons from initial perturbations of the form $\cos(\pi x)$ was examined for periodic (judging by the results obtained) boundary conditions. A detailed description of the results of analytical and numerical study of the Korteweg–de Vries equation in a wide range of governing parameters and the calculation algorithms can be found, for example, in [9–12]. A similar problem of existence of solitons on the surface of a thin layer of an incompressible non-Newtonian fluid was studied by Pumir et al. [13] using the qualitative and quantitative analysis of equations of a steady structure.

The objective of the present work is to study a flow of the type of a hydraulic shock on the free surface of a film of a viscoelastic liquid moving on an inclined plane in a wide range of Weber numbers and deviations of the Reynolds numbers from the critical value.

**Equations and Boundary Conditions.** We consider a two-dimensional motion of a layer of an incompressible viscoelastic liquid over a plane inclined at an angle $\alpha$ to the horizontal direction. The initial

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equations in tensor notation have the form

$$\rho \frac{dv_i}{dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}, \qquad \frac{\partial v_i}{\partial x_i} = 0,$$

where $\rho = \text{const}$ is the liquid density, $g_i$ are the components of the force of gravity, $v_i$ are the components of velocity, and $\tau_{ij}$ are the components of the stress tensor, which are written as

$$\tau_{ij} = -p\delta_{ij} + \eta\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) + \beta\Big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\Big)\Big(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k}\Big) + \gamma\Big(\frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\,\frac{\partial v_k}{\partial x_i}\,\frac{\partial v_k}{\partial x_j}\Big).$$

Here $p$ is the pressure, $\eta$ is the dynamic viscosity, $\beta$ and $\gamma$ are the characteristics of the liquid, $a_i = \partial v_i/\partial t + v_j\,\partial v_i/\partial x_j$ are the components of acceleration, and $(d/dt)_i = \partial v_i/\partial t + v_k \partial v_i/\partial x_k$; the subscripts acquire the values of 1, 2, and 3, and summation is performed over repeated subscripts.

We introduce Cartesian coordinates with the $x$ axis along the inclined plane and the $y$ axis across it and dimensionless variables choosing the following scales: the characteristic distances along and across the layer ($L_0$ and $H_0$, respectively), the time $t_0 = L_0/u_0$, $p_0 = \rho g H_0 \sin\alpha$, and $\tau_0 = \rho u_0^2$, where $u_0 = \rho g H_0^2 \sin\alpha/(2\eta)$ is the longitudinal velocity of a steady flow on the layer surface. The motion of the liquid considered here is described by the equations

$$u_x + v_y = 0, \qquad \frac{du}{dt} = \frac{gL_0}{u_0^2}\sin\alpha + (\tau_{xx})_x + \varepsilon^{-1}(\tau_{xy})_y,$$

$$\varepsilon\,\frac{dv}{dt} = -\frac{gL_0}{u_0^2}\cos\alpha + (\tau_{xy})_x + \varepsilon^{-1}(\tau_{yy})_y;$$

the components of the stress tensor are

$$\tau_{xx} = -p\,\frac{\sin\alpha}{\mathrm{Fr}} + \frac{2\varepsilon}{\mathrm{Re}}\,u_x + N[(u_y + \varepsilon^2 v_x)^2 + 4\varepsilon^2 u_x^2] - 2\varepsilon^2 M\Big[\Big(\frac{du}{dt}\Big)_x + u_x^2 + \varepsilon^2 v_x^2\Big],$$

$$\tau_{xy} = \frac{u_y + \varepsilon^2 v_x}{\mathrm{Re}} - \varepsilon M\Big[\Big(\frac{du}{dt}\Big)_y + 2u_x u_y + \varepsilon^2\Big(2v_x v_y + \Big(\frac{dv}{dt}\Big)_x\Big)\Big],$$

$$\tau_{yy} = -p\,\frac{\sin\alpha}{\mathrm{Fr}} + \frac{2\varepsilon}{\mathrm{Re}}\,v_y + N[(u_y + \varepsilon^2 v_x)^2 + 4\varepsilon^2 v_y^2] - 2M\Big[u_y^2 + \varepsilon^2\Big(v_y^2 + \Big(\frac{dv}{dt}\Big)_y\Big)\Big],$$

where $\varepsilon = H_0/L_0$, $\mathrm{Re} = u_0 H_0/\nu$, $N = \beta/(\rho H_0^2)$, $M = -\gamma/(\rho H_0^2)$, and $\mathrm{Fr} = u_0^2/(gH_0)$ ($\nu$ is the kinematic viscosity). We supplement these equations by the boundary conditions at the bottom and free surface of the liquid:

$$u = v = 0 \quad \text{for} \quad y = 0;$$

$$p_s = 0, \quad p_n = -\frac{\varepsilon^2 \mathrm{We}}{(1+\varepsilon^2 H_x^2)^{3/2}\sin\alpha}, \quad H_t + uH_x = v \ \text{ for } \ y = H(x,t),$$

where $p_s = \tau_{xy}\cos 2\varphi + (1/2)(\tau_{yy} - \tau_{xx})\sin 2\varphi$ and $p_n = \tau_{yy}\cos^2\varphi + \tau_{xx}\sin^2\varphi - \tau_{xy}\sin 2\varphi$ are the shear and normal to the free surface components of the stress tensor, $\tan\varphi = \varepsilon H_x$, $\mathrm{We} = \sigma/(\rho g H_0^2)$ is the Weber number, $\sigma$ is the surface tension, and $H(x,t)$ is a function that describes the shape of the free surface.

Assuming that the layer thickness is significantly smaller than the scale of longitudinal perturbations of the free surface and using the known procedure of decomposition of the equations and boundary conditions in the parameter $\varepsilon \ll 1$, Dandapat and Gupta [8] derived an equation for the free-surface shape with accuracy to terms of order $\varepsilon^3$. Note that it does not contain the constant $\beta$ because of the condition of incompressibility of the liquid. The surface tension in this equation is represented by a term proportional to $\varepsilon^3\,\mathrm{We}$. For high Weber numbers ($\mathrm{We} \sim O(\varepsilon^{-2})$), the effect of surface tension has the first order in the small parameter $\varepsilon$. With accuracy to terms of order $\varepsilon$, the equation for the free-surface shape investigated in [8] and in the present work acquires the form

$$H_t + 2H^2 H_x + (2/3)\varepsilon[A(H)H_x + B(H)H_{xxx}]_x = 0, \tag{1}$$

where

$$A(H) = [2(2/5)H^2 + M)\mathrm{Re}\,H - \cot\,\alpha]H^3, \qquad B(H) = \varepsilon^2\,\mathrm{We}\,H^3/\sin\alpha.$$

**Unsteady Solutions of the Model Equation.** According to Eq. (1), the shape of the free surface of a layer of a viscoelastic liquid changes due to nonlinear transfer with velocity $U = 2H^2$, nonlinear diffusion, which can be either positive or negative depending on the sign of $A(H)$, and a stabilizing action of surface tension forces. Linearizing (1) relative to the unperturbed level $H = 1 + h$ ($h \ll 1$) and substituting the solution in the form $h \sim \exp i[kx - (\omega_r + i\gamma)k]$, we obtain that the phase velocity of small perturbations $c = \omega_r/k = 2$ is independent of the wavenumber, and the growth rate of small perturbations is

$$\gamma = (2/3)\varepsilon k^2[2(2/5 + M)\mathrm{Re} - \mathrm{ctg}\,\alpha - \varepsilon^2 \mathrm{We}\, k^2/\sin\alpha]$$

($k$ is the wavenumber). It follows from this formula that small periodic perturbations are stable ($\gamma < 1$) or unstable ($\gamma > 1$) depending on whether the Reynolds number is greater or smaller than the critical value $\mathrm{Re}_* = 5\cot\alpha/(4 + 10M)$. The interval of wavenumbers of unstable perturbations is finite: $\Delta k = 0$–$k_*$,

$$k_*^2 = \frac{2(2 + 5M)(\mathrm{Re} - \mathrm{Re}_*)\sin\alpha}{5\varepsilon^2\mathrm{We}}.$$

The maximum value of the growth rate

$$\gamma_{\max} = \gamma(k_{\max}) = \frac{2(2 + 5M)^2(\mathrm{Re} - \mathrm{Re}_*)^2\sin\alpha}{75\varepsilon\,\mathrm{We}}$$

is reached for

$$k_{\max}^2 = \frac{(2 + 5M)(\mathrm{Re} - \mathrm{Re}_*)\sin\alpha}{5\varepsilon^2\,\mathrm{We}}.$$

As the viscoelasticity parameter $M$ increases, the critical Reynolds number decreases, and the growth rate and the interval of unstable wavenumbers increase. An increase in the Weber number does not change the critical Reynolds number, but the maximum growth rate decreases inversely proportionally to We and the interval of unstable wavenumbers to $\mathrm{We}^{-1/2}$.

To study the evolution of finite-amplitude perturbations, numerical methods should be used. Equation (1) was solved using an explicit finite-difference scheme in which the transfer was approximated by an upstream one-sided difference, since the coefficient at the derivative $H_x$ equals $2H^2$ and is always positive, and the remaining terms were approximated by central differences

$$H1_i = H_i - 2H_i^2\,\frac{\delta t}{\delta x}\,(H_i - H_{i-1}) - \varepsilon\,\frac{\delta t}{\delta x^4}\,(a_i H_{i-2} - b_i H_{i-1} + c_i H_i - d_i H_{i+1} + e_i H_{i+2}), \qquad (2)$$

where $H1_i \equiv H_i^{n+1}$, $H_i \equiv H_i^n$, $\delta t$ is the time step, $\delta x$ is the coordinate step, $a_i = B_{i-1/2}$, $e_i = B_{i+1/2}$, $b_i = B_{i+1/2} + 3B_{i-1/2} - A_{i-1/2}\delta x^2$, $c_i = 3(B_{i+1/2} + B_{i-1/2}) - (A_{i+1/2} + A_{i-1/2})\delta x^2$, $d_i = 3B_{i+1/2} + B_{i-1/2} - A_{i+1/2}\delta x^2$, and $(A, B)_{i\pm1/2} = (1/2)[(A, B)_i + (A, B)_{i\pm1}]$.

Since the highest derivative in Eq. (1) has fourth order, the explicit scheme (2) is stable only if the condition $\delta t/\delta x^4 < 1$ is satisfied, which requires rather small time steps. At the same time, the use of some implicit scheme (for example, the Crank–Nicholson scheme) for increasing stability makes it necessary to solve large systems of algebraic equations for grid functions with a five-diagonal matrix. The standard method of five-point sweep in the case under consideration requires a restriction on the time step to ensure the diagonal prevailing of the matrix and, hence, the stability of the sweep. Though this restriction is less rigid than for the explicit scheme, the algorithm becomes more complicated; therefore, scheme (2) was chosen.

To trace the motion and evolution of the shape of the free surface of the hydraulic shock in a viscoelastic liquid, a constant level $H(0, t) = 1 + b$ ($b > 0$) is set at the left boundary of the computational domain $x = 0$–$x_{\max}$ and an unperturbed flow $H(x_{\max}, t) = 1$ is prescribed at the right boundary; in addition, the conditions $H_x = H_{xx} = 0$ are set at both boundaries of the computational domain. The initial perturbation was chosen in the form of a smoothed step. If the slope of the step is small, the main role in evolution of the perturbation belongs at first to the transfer of an elevation of the free surface with velocity $c = 2H^2$, which depends on the thickness of the liquid layer. Since $dc/dH > 0$, the velocity of sectors of the perturbation profile with a higher level of the liquid is greater, and the perturbation front becomes steeper with time. This growth of the profile steepness corresponds to the increase in the $x$ derivatives of the function $H(x, t)$; as a result, the contribution of terms that describe diffusion and the effect of surface tension increase.

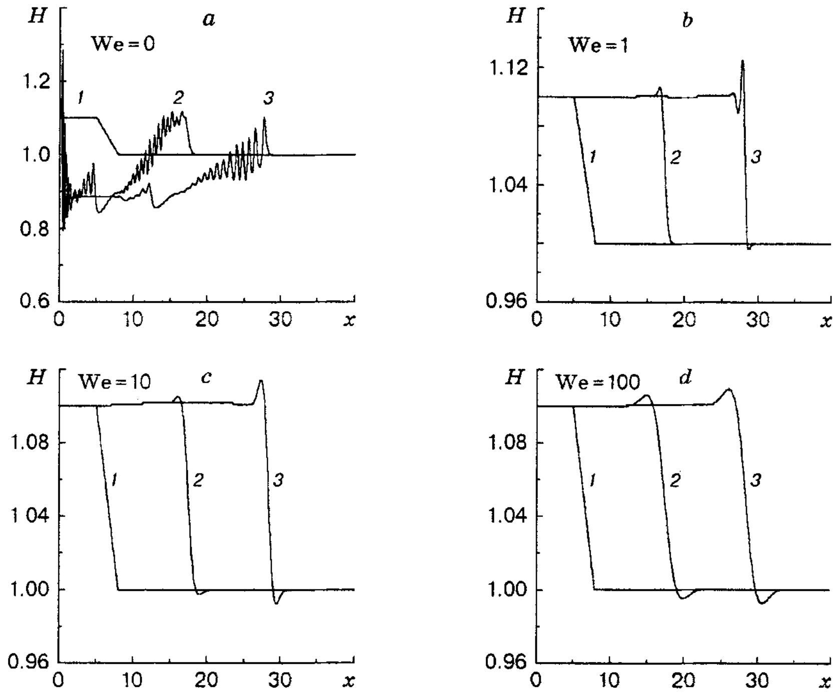


Fig. 1

For Re < Re$_*$, the width of the perturbation front increases, and a smoothed shock of constant width can appear. The calculations show that the perturbation profile is monotonic for low surface tension. We also note that the front contains a certain "foot" extended downstream, which is a consequence of the nonlinear character of diffusion. If the surface tension is rather large (We $\sim \varepsilon^{-2}$), a smoothed shock with a constant front width is still gradually formed, but perturbations appear on the profile upstream and downstream of the point of maximum steepness. Thus, the height of the free surface in a small vicinity of the shock zone is greater in the upstream region than in the perturbed part of the flow far behind the front, and in the downstream region it is smaller than in the free stream far ahead of the front.

For Re > Re$_*$, instability of the spatial structure of the shock arises, but this instability can be stabilized by surface-tension forces. Figure 1 shows the calculated profiles of the free surface of a viscoelastic liquid for Re = 2.5 > Re$_*$ and $M = 0.1$ for several values of the Weber number (curves 1–3 correspond to the free-surface shape at times $t = 0$, 5, and 10). When there is no surface tension (Fig. 1a), the flow is strongly unstable, and the profile shape has a clear saw-tooth character. We note that this shape of the profile is a consequence of physical rather than numerical instability, since each oscillation contains at least ten points of the finite-difference grid, and the number of time steps is about a hundred thousand. The effect of surface tension is manifested in stabilization of instability and noticeable smoothing of the profile of the propagating perturbation. The layer thickness in the region behind the shock remains almost the same. Near the shock, when approaching it from the upstream region, spatial oscillations are formed on the profile, and the amplitude of the first oscillation is not small (Fig. 1b). The slope of the front of the smoothed shock is greater than the slope of the profile of the initial perturbation. Ahead of the front, when approaching it from the free stream, a small region is formed, where the level of the liquid is lower than in the unperturbed stream. As the Weber number increases (Fig. 1c and d), the nonmonotonic structure of the disturbance front is retained. The width of the oscillation region increases, the amplitude of oscillations behind the front decreases, and the "valley" in the liquid surface ahead of the front increases. The slope of the free-surface profile in the shock region decreases when passing to high Weber numbers.

We also calculated the evolution of perturbations of the above-mentioned shape with increasing

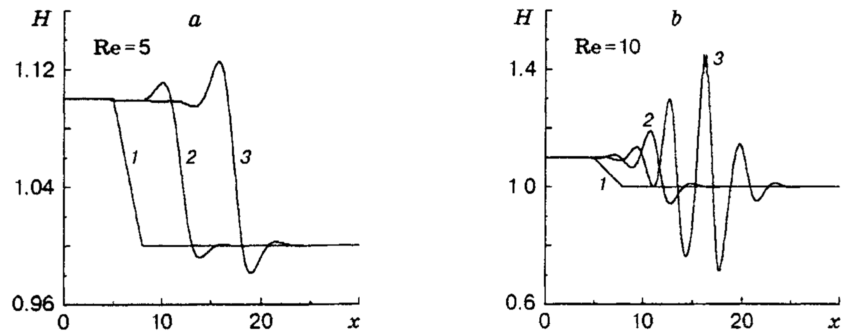


Fig. 2

Reynolds number. Figure 2 shows the results corresponding to the viscoelasticity parameter $M = 0.1$, Weber number We = 100, and Reynolds numbers Re = 5 (a) and 10 (b). A comparison of the profiles of the free surface of the liquid for Re = 2.5 and 5 shows that, as the Reynolds number increases, a growth in the surface elevation behind the shock and a greater depth of the liquid ahead of the shock are observed. In addition, when moving to the region ahead of the shock, the layer thickness becomes more unperturbed after the decrease mentioned and acquires the value $H = 1$. The slope of the front increases. It follows from Fig. 2b that the free surface acquires the form of smooth spatial oscillations of high amplitude: $H_{\max} \approx 1.45$ and $H_{\min} \approx 0.7$. In the course of time, the perturbed region becomes similar to a set of "solitons" obtained in [8] by solving the Korteweg–de Vries equation with periodic initial data and boundary conditions, the amplitudes of these solitons increase monotonically, and at the time $t = 10$, the maximum thickness of the layer in the first oscillation is $H_{\max} \approx 1.4$, and the minimum thickness of the layer is $H_{\min} \approx 0.5$.

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